DIFFERENTIAL CALCULUS

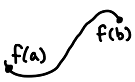
! use radians   ! use 3 decimal places for answers and 6 while solving

NOTATION:

Leibniz:	$\frac{dy}{dx}$ or $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$	$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$	- can specify variables for differentiation
Lagrange:	$f'(x)$ or y'	y''	- faster

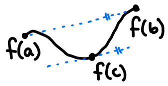
L'HÔPITAL'S RULE: if $\lim\limits_{x\to n}\frac{f(x)}{g(x)}$ is indeterminate, then $\lim\limits_{x\to n}\frac{f(x)}{g(x)} = \lim\limits_{x\to n}\frac{f'(x)}{g'(x)}$ (can be repeated)

INTERMEDIATE VALUE THEOREM (IVT): if a function is continuous b/w a and b, it takes on every value between f(a) and f(b)



its derivative also takes on every value between f'(a) and f'(b)

MEAN VALUE THEOREM (MVT): if f(x) is differentiable b/w [a,b], then $\ni c$ where $a<c<b$ such that



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

POWER RULE: $y = ax^n$, $y' = nax^{n-1}$

CHAIN RULE: $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ if $y = f(u)$ & $u = g(x)$
or $\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$

POWER CHAIN RULE: $\frac{d}{dx}f(a) = f'(a)\frac{da}{dx} \longrightarrow \frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

PRODUCT RULE: $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$

QUOTIENT RULE: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

TRIG:
$$\frac{d}{dx}\left\{\begin{array}{l} \sin u = \cos u \\ \cos u = -\sin u \\ \tan u = \sec^2 u \\ \cot u = -\csc^2 u \\ \sec u = \sec u \tan u \\ \csc u = -\csc u \cot u \end{array}\right\}\frac{du}{dx}$$

INVERSE TRIG:
$$\frac{d}{dx}\left\{\begin{array}{l} \sin^{-1} u = \sqrt{1-u^2}^{\,-1} \\ \cos^{-1} u = -\sqrt{1-u^2}^{\,-1} \\ \tan^{-1} u = (1+u^2)^{-1} \\ \cot^{-1} u = -(1+u^2)^{-1} \\ \sec^{-1} u = \left(|u|\sqrt{u^2-1}\right)^{-1} \\ \csc^{-1} u = -\left(|u|\sqrt{u^2-1}\right)^{-1} \end{array}\right\}\frac{du}{dx}$$

$\sin^{-1}(u)\begin{cases} D: [-1,1] \\ R: [-\pi/2, \pi/2] \end{cases}$

$\csc^{-1}(u)$ $D: (\infty,-1][1,\infty)$

EXP & LOG:
$$\frac{d}{dx}\left\{\begin{array}{ll} e^u = e^u\frac{du}{dx} & \\ a^u = \ln(a)\cdot a^u\frac{du}{dx} & a\geq 0 \\ \ln(u) = \frac{1}{u}\cdot\frac{du}{dx} & u>0 \\ \log_a(u) = \frac{1}{\ln(a)\cdot u}\cdot\frac{du}{dx} & a>0, a\neq 1 \end{array}\right.$$

$\frac{\ln(u)}{\ln(a)} = \frac{d}{dx}\frac{\ln(u)}{\ln(a)} = \frac{1}{\ln(a)}\frac{1}{u}\frac{du}{dx}$

IMPLICIT DIFFERENTIATION: treat y as an equation & differentiate both sides:

ex: $x^2 + y^2 = 1$
$2x + 2y(y') = 0$
$y' = -x\cdot y^{-1}$

$y'' = -1\cdot y^{-1} - x\cdot y^{-2}y'$
$= -\frac{1}{y} - \frac{x}{y^2}\cdot\frac{-x}{y}$
$y'' = \frac{-x^2-1}{y^3}$

taking the second derivative

ex: $x = t^2 + t$, $y = \sin(t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{2t+1}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\cdot\frac{\cos(t)}{2t+1}\cdot\frac{1}{2t+1}$

$\frac{d^2y}{dx^2} = \frac{-\sin(t)(2t+1) - 2\cos(t)}{(2t+1)^3}$

$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt}\cdot\frac{dy}{dx}\cdot\frac{dt}{dx}$

INVERSE IDENTITY: if f(x) and g(x) are inverse functions, then $g'(x) = \frac{1}{f'g(x)}$

L'HÔPITAL'S RULE: if $\lim\limits_{x\to n}\frac{f(x)}{g(x)}$ is $\underbrace{\text{indeterminate}}_{\frac{0}{0},\ \frac{\infty}{\infty},\ \infty\cdot 0,\ \infty-\infty,\ 1^\infty,\ 0^0,\ \infty^0}$, then $\lim\limits_{x\to n}\frac{f(x)}{g(x)} = \lim\limits_{x\to n}\frac{f'(x)}{g'(x)}$

ex. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sec(x)}{1+\tan(x)}$ $x\to\frac{\pi}{2}^+\ \frac{\infty}{\infty}$ $x\to\frac{\pi}{2}^-\ \frac{-\infty}{-\infty}$

LHR $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sec(x)\tan(x)}{\sec^2(x)} = \frac{\tan(x)}{\sec(x)}$

$= \lim\limits_{x\to\frac{\pi}{2}}\sin(x) = 1$

ex. $\lim\limits_{x\to\infty}x\cdot\sin\left(\frac{1}{x}\right) = \infty\cdot 0$

$\lim\limits_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{x^{-1}} = \frac{0}{0}$

LHR $\lim\limits_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)(-x^{-2})}{-x^{-2}}$

$= \lim\limits_{x\to\infty}\cos\left(\frac{1}{x}\right) = 1$

ex. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \infty - \infty$

$\lim\limits_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} = \frac{0}{0}$

LHR $\lim\limits_{x\to 1}\frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}} = \frac{x-1}{x\ln x + x - 1}$

LHR $\lim\limits_{x\to 1}\frac{1}{\ln x + 1 + 1} = \frac{1}{2}$

if $\lim\limits_{x\to a}\ln f(n) = L$, then $\lim\limits_{x\to a}f(x) = \lim\limits_{x\to a}e^{\ln f(x)} = L$

ex. $f(x) = \left(1+\frac{1}{x}\right)^x$, $\lim\limits_{x\to\infty}f(x) = 1^\infty$

$\lim\limits_{x\to\infty}\ln(f(x)) = \lim\limits_{x\to\infty}x\ln\left(1+\frac{1}{x}\right) = \infty\cdot 0$

$\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = \frac{0}{0}$

LHR $\lim\limits_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim\limits_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$

$\lim\limits_{x\to\infty}f(x) = \lim\limits_{x\to\infty}e^1 = e$

ex. $f(x) = x^x$, $\lim\limits_{x\to 0^+}f(x) = 0^0$

$\lim\limits_{x\to 0^+}\ln(f(x)) = \lim\limits_{x\to 0^+}x\ln(x) = 0\cdot\infty$

$\lim\limits_{x\to 0^+}\frac{\ln(x)}{x^{-1}} = \frac{\infty}{\infty}$

$\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0^+}-x = 0 = \lim\limits_{x\to 0^+}\ln(f(x))$

$\lim\limits_{x\to 0^+}f(x) = \lim\limits_{x\to 0^+}e^0 = 1$

APPLICATIONS OF DERIVATIVES & GRAPHIC ANALYSIS:

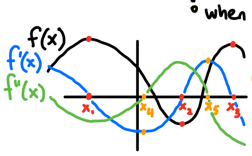
critical point: $\frac{d}{dx}$ is 0/doesn't exist

! critical points must be defined in the original function

$f'(x)$: + x_1 − x_2 + x_3 − x_4 +
max   min   ...   ...

! when f(x) is concave up, $f''(x) > 0$;
when f(x) is concave down, $f''(x) < 0$

! establish continuity and consider endpoints



inflection point: change in concavity in f(x); $f''(x) = 0$

if $f'(x) = 0$: $f''(x) > 0$ = local min; $f''(x) < 0$: local max

LINEARIZATION: tangent $L(x) = f(a) + f'(a)(x-a)$ is linearization of f

$f(x) \approx L(x)$ is the standard linear approximation of f

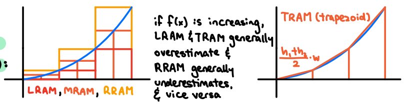
$x = a$ is the center of the approximation

DEFINITIONS: $\frac{df}{dx} = \lim\limits_{b\to 0}\frac{f(x+b) - f(x)}{b}$

$\int_a^b f(x)dx = \lim\limits_{n\to\infty}\sum_{i=1}^{n}f(c_i)\delta x$ where $\delta x = \frac{b-a}{n}$

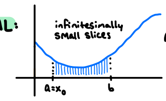
INTEGRAL CALCULUS

RAM – RECTANGLE APPROXIMATION METHOD:



if $f(x)$ is increasing, LRAM & TRAM generally overestimate & RRAM generally underestimates, & vice versa

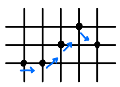
DEFINITE INTEGRAL:



$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

SLOPE-FIELD ANALYSIS – EULER'S METHOD: estimating given $\frac{dy}{dx}$ and an initial point (x,y)

overestimates when convex, underestimates when concave



e.g. $\frac{dy}{dx}=x+y$ and $f(2)=0$, $f(3)=?$

$\frac{3-2}{5}=.2=\Delta x$, $\Delta x\cdot\frac{dy}{dx}=\Delta y$

↳ can decide # of steps used

x	y	$\frac{dy}{dx}$	Δx	$\Delta y=\frac{dy}{dx}\Delta x$
2	0	2	.2	.4
2.2	.4	2.6	.2	.52
2.4	.92	3.32	.2	.664
2.6	1.584			
⋮	⋮	⋮	⋮	⋮

SOME BASIC RULES: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

$\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ $\qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)$$

MVT (INTEGRALS): if $f(x)$ is continuous from $[a,b]$, there is a c $a\le c\le b$ such that $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$ where $f(c)$ is the average value of $f(x)$ in $[a,b]$

AREA BETWEEN CURVES: if $f(x)$ and $g(x)$ are continuous $[a,b]$ and $f(x)\ge g(x)$ $[a,b]$, then the area between the curves

$$=\int_a^b [f(x)-g(x)]\,dx$$

if $g(x)$ becomes greater at $x=c$ then the area

$$=\int_a^c [f(x)-g(x)]\,dx + \int_c^b [g(x)-f(x)]\,dx \quad (\text{\& so on})$$

can also be done in terms of y by slightly transforming the functions

KINEMATICS: if $s(t)$ is position, $\frac{ds}{dt}$ = velocity $v(t)$ and

$\int_a^b v(t)\,dt$ = displacement from $t=a$ to b, $= s(b)-s(a)$

$\int_a^b |v(t)|\,dt$ = total distance traveled (use calculator)

VOLUME: Shapes tested: circle, semicircle, square, rectangle, equilateral triangle, right triangle

e.g. a solid enclosed in $x^2+y^2=1$, the cross sections of which are perpendicular to the x-axis and represent diameters of circles

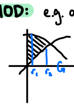
↳ area equation: $A(x)=\pi r^2$

↳ r in terms of x: $y^2=1-x^2$, $y=\pm\sqrt{1-x^2}=r$

→ plug in: $A(x)=\pi\left(\sqrt{1-x^2}\right)^2$, volume $=\int_{-1}^{1}\pi(1-x^2)\,dx$

$$=2\pi\int_0^1(1-x^2)\,dx = 2\pi\left(x-\frac{x^3}{3}\right)\Big|_0^1 = \frac{4\pi}{3}$$

WASHER METHOD: e.g. area enclosed by $x=0$, $y=\sin(x)$, $y=\cos(x)$ is rotated around the x-axis



$A=\pi\int_a^b (r_1)^2\,dx - \int_a^b (r_2)^2\,dx = \int_a^b (r_1^2-r_2^2)\,dx$, $r_1>r_2$ $\quad r_1=\cos(x)$, $r_2=\sin(x)$ $\quad \cos(x)>\sin(x), x=[0,\frac{\pi}{4}]$

$$=\pi\int_0^{\pi/4}(\cos^2x-\sin^2x)\,dx = \pi\int_0^{\pi/4}\cos 2x\,dx = \pi\frac{\sin 2x}{2}\Big|_0^{\pi/4}$$

$$=\pi\left[\frac{\sin(\pi/2)}{2}-\frac{\sin(0)}{2}\right]=\frac{\pi}{2}$$

ARC LENGTH: $L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if y is a smooth function of x on (a,b)

IMPROPER INTEGRALS: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$

e.g. $\int_{-1}^{\infty} xe^{x}\,dx = \lim_{b\to\infty}\int_{-1}^{b} xe^{x}\,dx = \lim_{b\to\infty} -xe^{-x}\Big|_1^b$ $\qquad$! use limit notation in FRQs

$$=\lim_{b\to\infty}\frac{-b-1}{e^b}+\frac{2}{e} \overset{\text{LHR}}{=} \lim_{b\to\infty}\frac{-1}{e^b}+\frac{2}{e} = 0+\frac{2}{e}=\frac{2}{e}$$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$ $\qquad \int_1^\infty \frac{1}{x^n}\,dx = \frac{1}{n-1}$ for $n>1$

FUNDAMENTAL THEOREM OF CALCULUS: if $f(x)$ is continuous from $[a,b]$, differentiable for x from $[a,b]$, and we let $F(x)=\int_a^x f(t)\,dt$, then

$\frac{d}{dx}F(x)=f(x)$ $\quad$ e.g. $\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{1+e^t}\,dt = \frac{d}{dx}\left(\int_0^{x^2=u_1}\frac{1}{1+e^t}\,dt - \int_0^{2x=u_2}\frac{1}{1+e^t}\,dt\right) = \frac{1}{1+e^{u_1}}\frac{du_1}{dx} - \frac{1}{1+e^{u_2}}\frac{du_2}{dx}$

FTC pt. 2: $\int_a^b f(x)\,dx = F(b)-F(a)$ where $F(x)$ is the anti-derivative of $f(x)$

anti-deriving thought process: algebra → trig → basics → substitution → parts

TRIG:
$$\int\begin{Bmatrix}\cos(u)\\ \sin(u)\\ \sec^2(u)\\ \csc^2(u)\\ \sec(u)\tan(u)\\ \csc(u)\cot(u)\\ \tan(u)\\ \sec(u)\end{Bmatrix}du = \begin{Bmatrix}\sin(u)\\ -\cos(u)\\ \tan(u)\\ -\cot(u)\\ \sec(u)\\ -\csc(u)\\ \ln|\sec(u)|\\ \ln|\sec(u)+\tan(u)|\end{Bmatrix}+C$$

$\ln|\sec(u)| = -\ln|\cos u|$

EXP & LOG:
$$\int\begin{Bmatrix}e^u\\ a^u\\ \ln(u)\\ \log_a(u)\end{Bmatrix}du = \begin{Bmatrix}e^u\\ \frac{a^u}{\ln a}\\ u\ln(u)-u\\ \frac{u\ln(u)-u}{\ln a}\end{Bmatrix}+C$$

$\int\frac{1}{u}\,du = \ln|u|+c$

ANTI-POWER RULE: $F(x)=x^n \Rightarrow f(x)=\frac{x^{n+1}}{n+1}+c$; $\int u^{-1}\,du=\ln|u|+c$

e.g. $\int_{-1}^{3}(x^3+1)\,dx = \left[\frac{x^4}{4}+x\right]\Big|_{-1}^{3} = \left(\frac{3^4}{4}+3\right)-\left(\frac{-1^4}{4}+(-1)\right)=24$ $\quad$ also using FTC pt. 2

e.g. $\int_{\pi/6}^{5\pi/6}\csc^2(x)\,dx = [-\cot(x)]\Big|_{\pi/6}^{5\pi/6} = -\cot\frac{5\pi}{6}+\cot\frac{\pi}{6}=2\sqrt{3}$

e.g. $\int_{-1}^{2}3^x\,dx = \left[\frac{3^x}{\ln 3}\right]\Big|_{-1}^{2} = \frac{3^2}{\ln 3}-\frac{3^{-1}}{\ln 3}=\frac{26}{3\ln 3}$

"reverse chain rule"

U-SUBSTITUTION: e.g. $\int\sin(x)e^{\cos x}\,dx$, let $u=\cos(x)$ $\quad \frac{du}{dx}=-\sin(x)$, $dx=\frac{-du}{\sin(x)}$

$=\int e^u\sin(x)\,dx$

$=\int e^u\cdot -du = -\int e^u\,du$

$=-e^u+c$

$=-e^{\cos x}+c$

e.g. $\int x^2\sqrt{5+2x^3}\,dx$ let $u=5+2x^3$, $du=6x^2\,dx$

$=\int x^2u^{\frac{1}{2}}\,dx = \int u^{\frac{1}{2}}x^2\,dx$

$=\int u^{\frac{1}{2}}\frac{du}{6} = \frac{1}{6}\int u^{\frac{1}{2}}\,du = \frac{1}{6}\cdot\frac{2}{3}u^{\frac{3}{2}}+c$

$=\frac{1}{9}\left(5+2x^3\right)+c$

e.g. $\int_0^{\pi}\tan(x)\sec^2(x)\,dx$, let $u=\tan(x)$, $du=\sec^2(x)\,dx$

$=\int_0^{\pi/3}u\cdot du$ $\quad \begin{Bmatrix}x=0,\ u=0\\ x=\pi/3,\ u=\sqrt{3}\end{Bmatrix}$ using FTC

$=\left[\frac{u^2}{2}\right]\Big|_0^{\sqrt{3}} = \left[\frac{\sqrt{3}^2}{2}\right]-\left[\frac{0^2}{2}\right]=\frac{3}{2}$

e.g. $\int_0^1\frac{x}{x^2-4}\,dx$ $\quad$ let $u=x^2-4$, $du=2x\,dx$

$=\int_0^1\frac{1}{u}\cdot x\,dx = \frac{1}{2}\int_0^1\frac{1}{u}\,du$ $\quad \begin{Bmatrix}x=0,\ u=-4\\ x=1,\ u=-3\end{Bmatrix}$

$=\frac{1}{2}\int_{-4}^{-3}\frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]\Big|_{-4}^{-3} = \frac{1}{2}\left[\ln(4)-\ln(3)\right]$

INTEGRATION BY PARTS: $\int f(x)g'(x)\,dx = f(x)g(x)-\int f'(x)g(x)\,dx$

$\int u\,dv = uv-\int v\,du$

LIPET: ln, inverse trig, polynomial, exponential, trig for u

e.g. $\int e^x\cdot\cos(x)\,dx = \int u\,dv = uv-\int v\,du$ $\quad u=\cos(x)$, $du=-\sin(x)\,dx$; $v=e^x$, $dv=e^x\,dx$

$=e^x\cos(x)-\boxed{\int e^x\sin(x)\,dx}+\underline{e^x\sin(x)-\int e^x\cos(x)\,dx}$

repeat $\quad u=\sin(x)$, $du=\cos(x)\,dx$; $v=e^x$, $dv=e^x\,dx$

$\int e^x\cdot\cos(x)\,dx = \frac{1}{2}\left[e^x\cos(x)+e^x\sin(x)\right]+c$

TABULAR METHOD: e.g. $\int x^3\cdot\cos(x)\,dx = x^3\sin(x)+3x^2\cos(x)-6x\sin(x)-6\cos(x)+C$

u :	x^3	$3x^2$	$6x$	6	0 ← u = terminating part
	+	−	+	−	
dv :	$\cos(x)$	$\sin(x)$	$-\cos(x)$	$-\sin(x)$	$\cos(x)$ ← v = easily integratable

SEPARATION BY PARTS:

1: separate variables: $\frac{dy}{dx}=x^2y^2 \longrightarrow y^{-2}\,dy = x^2\,dx$

2: anti-derive: $\int y^{-2}\,dy = \int x^2\,dx \longrightarrow -y^{-1}=\frac{x^3}{3}+c$

3: solve for c: $-(1)^{-1}=\frac{(1)^3}{3}+c \longrightarrow c=\frac{-4}{3}$

4: find y: $\frac{-1}{y}=\frac{x^3}{3}-\frac{4}{3}$ $\quad y=\frac{3}{4-x^3}$

SEQUENCES & SERIES

SEQUENCE: list of values, described explicitly or recursively

↳ **ARITHMETIC:** common difference; $a_n = a_1 + (n-1)d$; always diverges

↳ **GEOMETRIC:** common ratio; $a_n = a_1 \cdot r^{n-1}$

SERIES: sum of a sequence; could be finite or infinite

$$S_n = \frac{a_1(r^n-1)}{r-1}$$

$$S_n = \frac{a_1}{1-r} \quad |r|<1 \qquad \sum_{k=0}^{\infty}\left(\frac{1}{n}\right)^k = \frac{1}{n-1}$$

→ if $\lim_{n\to\infty}\sum_{k=1}^{n} a_k = Q$ where Q is finite, the series converges

else it diverges

e.g. 1+1-1+1-1+1⋯ diverges (doesn't come to a distinct value)

if $\lim_{n\to\infty}\sum_{k=1}^{n} a_k = Q$ where Q is finite, $\lim_{k\to\infty} a_k = 0$

↳ **POWER:** $\sum_{n=0}^{\infty} c_n(x-a)^n$ $\quad c_n$ can be any real value

→ a is the center (typically 0)

→ geometric series with $r=x$

e.g. $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots+x^n = \frac{1}{1-x}$

10.1 e.g. $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+\cdots+(-1)^n x^n = \sum_{n=0}^{\infty} -x^n \quad |-x| = |x| < 1$

e.g. $\frac{1}{1-2x} = 1+2x+(2x)^2+\cdots+2^n x^n = \sum_{n=0}^{\infty} 2^n x^n \quad |2x|<1 \quad x \in (-\frac{1}{2}, \frac{1}{2})$ interval is smaller

e.g. $\frac{1}{x} = \frac{1}{\underbrace{1+(x-1)}_{\text{obtain } x-a \text{ format}}} = 1-(x-1)+(x-1)^2-\cdots+(-1)^n(x-1)^n = \sum_{n=0}^{\infty}(-1)^n \underbrace{(x-1)^n}_{\text{center at } x=1} \quad |x-1|<1;\ x\in(0,2)$

e.g. $\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}0 + \frac{d}{dx}x + \cdots = 0+1+2x+3x^2+\cdots+nx^{n-1} = \sum_{n=0}^{\infty} nx^{n-1}$

e.g. $\ln(1+x) = \int \frac{1}{1+x}dt = \int 1dt + \int -xdt + \int(-x)^2 + \cdots = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$

→ if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ where $|x-a|<R$ converges, then

$\frac{d}{dx}f(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$ converges at $|x-a|<R$, and

$\int f(x)dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$ converges at $|x-a|<R$

10.2 **TAYLOR SERIES:** Series generated at $x=a$ such that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^n(a)(x-a)^n}{n!} = \sum_{k=0}^{\infty}\frac{f^k(a)(x-a)^k}{k!}$$

e.g. $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$; $P(0)=5$; $P'(0)=7$; $P''(0)=11$; $P^3(0)=13$; $P^4(0)=17\cdots$

$P(0) = a_0 + 0 + 0 + \cdots;\ a_0 = 5 \qquad a_n = \frac{P^n(0)}{n!}$

$P'(0) = 0 + a_1 + 0 + \cdots;\ a_1 = 7$

$P''(0) = 0 + 0 + 2a_2 + 0 + \cdots;\ a_2 = \frac{11}{2}$

e.g. $\cos(x)$ at $x=0$ $\quad \cos'(0)=0;\ a_1=0 \ \cdots \frac{(-1)^n x^{2n}}{(2n)!}$ oscillating b/w +, - ; skips terms where $f^n(0)=0$

$\cos(0)=1;\ a_0=1 \quad \cos''(0)=-1;\ a_2=-\frac{1}{2}$

nonzero center → e.g. e^x at $x=2$; $f(x)=f'(x)=\cdots=f^n(x)=e^2$; $e^x = \sum_{k=0}^{\infty}\frac{e^2(x-2)^k}{k!}$ ($f^k(a)$)

→ "nth order Taylor Series" = # of derivations

→ "at x = a" = centered at a

↳ **MACLAURIN SERIES:** Taylor Series centered at $x=0$

→ $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ $\qquad$ → $\ln(1+x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{k+1} \quad -1<x\leq 1$

→ $\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$ $\qquad$ → $\tan^{-1}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} \quad |x|\leq 1$

→ $\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

e.g. expansion of $\cos(t^5)$ $\quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$

$\cos(x^5) = 1 - \frac{(x^5)^2}{2!} + \frac{(x^5)^4}{4!} - \cdots + \frac{(-1)^n x^{5 \cdot 2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{10n}}{(2n)!}$

10.3 **TAYLOR'S THEOREM:** $P_n(x) = \sum_{k=0}^{n}\frac{f^n(a)}{n!}(x-a)^n$, $f(x) = P_n(x) + R_n(x)$ then

↳ **ERROR TERM:** $R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \quad a\leq c\leq x$

if $R_n(x)\to 0$ as $n\to\infty$ for $x\in I$, $P(x)$ at $x=a$ converges to f on I

factorial grows faster than exponential

↳ **REMAINDER ESTIMATION:** (remainder bounding) if there are positive constants M and r such that

$|f^{(n+1)}(t)| \leq Mr^{n+1}$ for all $a<t<x$, then

$$|R_n(x)| \leq M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}$$ (r = 1 for Maclaurin series)

e.g. $e^{7x} = \sum_{n=0}^{\infty}\frac{(7x)^n}{n!}$ $\quad M = e^{7x}$

$f(t) = e^{7t} \to f^{n+1}(t) = 7^{n+1}e^{7t}$

for $[0,x]$ $|f^{n+1}(t)| \leq M\cdot 7^{n+1}$ $0\leq t\leq x$

for $[x,0]$ $|f^{n+1}(t)| \leq M\cdot 7^{n+1}$ $x\leq t\leq 0$

so $R_n(x)\to 0$ as $n\to\infty$

10.4 **RADIUS OF CONVERGENCE:** $\sum_{n=0}^{\infty} c_n(x-a)^n$

converges when $|x-a|<R$

converges for all x $R=\infty$

converges at $x=a$ $R=0$ (diverges elsewhere)

↳ **n-TH TERM TEST:** $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ doesn't exist or $\neq 0$

e.g. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, $\lim_{b\to\infty}(-1)^b x^{2b} \neq 0$, series diverges ↳ (no vice versa)

↳ **DIRECT COMPARISON TEST:** if $\sum c_N$ converges, a series $\sum a_n$ with nonzero terms

- converges if $a_n \leq c_N$ for some $n<N$ "falls behind"
- diverges if $a_n \geq c_N$ for some $n<N$ "catches up"

e.g. $\sum\frac{x^{2n}}{(n!)^2} \leq \sum\frac{(x^2)^n}{n!} = e^{x^2}$ where $R=\infty$

↳ **ABSOLUTE VALUE TEST:** $\sum a_n$ converges if $\sum|a_n|$ converges

e.g. $\sum_{n=0}^{\infty}\frac{\sin(x)^n}{n!} < \sum_{n=0}^{\infty}\frac{|\sin(x)|^n}{n!} < \sum_{n=0}^{\infty}\frac{x^n}{n!} = e$

RATIO TEST: let $\sum a_n$ be a series of pos. terms

if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ < 1: converges

> 1: diverges

$= 1$: ☹

r.o.c. is $|x|<10$

e.g. $\sum_{n=0}^{\infty}\frac{nx^n}{10^n} \to \lim_{n\to\infty}\frac{(n+1)|x^{n+1}|}{10^{n+1}}\cdot\frac{10^n}{n|x^n|} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{|x|}{10} = \frac{|x|}{10}$

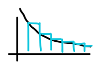
10.5 ENDPOINT TESTS:

↳ **INTEGRAL TEST:** if a_n is always positive and is decreasing from some N,

$\sum_{N}^{\infty} a_n$ and $\int_{N}^{\infty} f(x)dx$ where $a_n = f(n)$ both converge or diverge

e.g. $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} \to \lim_{b\to\infty}\int_1^b x^{-\frac{3}{2}}dx = \lim_{b\to\infty}\left[-2x^{-\frac{1}{2}}\right]_1^b = 2$, so $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ converges

! $\sum^{\infty} a_n \geq \int^{\infty} f(x)dx$



↳ **P-SERIES TEST:** $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p>1$, and otherwise diverges

LIMIT COMPARISON TEST: if $a_n>0$ and $b_n>0$ for all $n\geq N$,

if $\lim_{n\to\infty}\frac{a_n}{b_n} = c$, $0<c<\infty$, $\sum a_n$ & $\sum b_n$ both converge or diverge

if $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, $\sum a_n$ converges

if $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, $\sum a_n$ diverges

e.g. $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2} \to a_n$, $b_n = \frac{1}{n}$, $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{2n^2+n}{n^2+2n+1} = 2$; since $\sum b_n$ diverges $\sum a_n$ diverges

ALTERNATING SERIES TEST: (Leibniz's Theorem) $\sum_{n=1}^{\infty}(-1)^n a_n$ converges if $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing eventually

! a_n must ≥ 0

alternating series: largest error is 1st unused term

PARAMETRIC FUNCTIONS:

VECTORS: $r(t) = \langle x(t), y(t)\rangle$ $\quad$ displacement $= \langle\int_a^b v_x(t)dt, \int_a^b v_y(t)dt\rangle$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $\qquad v(t) = \langle\frac{dx}{dt}, \frac{dy}{dt}\rangle$ $\qquad$ distance $= \int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$

$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$ $\qquad a(t) = \langle\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\rangle$

tdt = total distance traveled $\qquad$ speed $|v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

POLAR COORDINATES: $(x,y)\to(r,\theta)$ $\quad x = r\cos\theta$, $y = r\sin\theta$; $x^2+y^2=r^2$ $\tan\theta = \frac{y}{x}$

e.g. $(2,2)\to(2\sqrt{2}, \pi/4)$

e.g. $r = 4\sin\theta$

$r^2 = 4r\sin\theta = 4y$

$r^2 = 4y = x^2+y^2$

$x^2+y^2-4y=0$

$x^2+(y-2)^2=4$

$r=2$, $C=(0,2)$

e.g. $r = 2\sin(3\theta)$

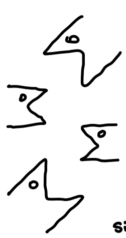
slope at $\theta = \pi/6$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ where $t=\theta$

$y = 2\sin(3\theta)\sin\theta$

$x = 2\sin(3\theta)\cos\theta$

$\frac{dy}{dx} = \frac{6\cos3\theta\sin\theta + \cos\theta\, 2\sin3\theta}{6\cos3\theta\cos\theta - \sin\theta\, 2\sin3\theta} = -\sqrt{3}$



$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

$$A_{between} = \frac{1}{2}\int_\alpha^\beta(r_1^2 - r_2^2)d\theta$$

e.g. area of $r = 2(1+\cos\theta) = \int_{\theta=0}^{\theta=2\pi}\frac{1}{2}r^2d\theta = \int_0^{2\pi}\frac{1}{2}4(1+\cos\theta)^2d\theta$

$= \int_0^{2\pi}2(1+2\cos\theta+\cos^2\theta)d\theta = \int_0^{2\pi}(2+4\cos\theta+2\frac{1+\cos2\theta}{2})d\theta$

$= \int_0^{2\pi}(3+4\cos\theta+\cos2\theta)d\theta = \left[3\theta+4\sin\theta+\frac{\sin2\theta}{2}\right]_0^{2\pi} = 6\pi$

$\sinh: \frac{e^x - e^{-x}}{2}$

$\cosh: \frac{e^x + e^{-x}}{2}$